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Multipole invariants and non-Gaussianity

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ABSTRACT

We propose a framework for separating the information contained in the CMB multipoles, $a_{\ell m}$, into its algebraically independent components. Thus we cleanly separate information pertaining to the power spectrum, non-Gaussianity and preferred axis effects. The formalism builds upon the recently proposed multipole vectors (Copi, Huterer & Starkman 2003; Schwarz & al 2004; Katz & Weeks 2004), and we elucidate a few features regarding these vectors, namely their lack of statistical independence for a Gaussian random process. In a few cases we explicitly relate our proposed invariants to components of the n-point correlation function (power spectrum, bispectrum). We find the invariants' distributions using a mixture of analytical and numerical methods. We also evaluate them for the co-added WMAP first year map.

Key words: cosmic microwave background - Gaussianity tests - statistical isotropy.

1 INTRODUCTION

The remarkable quality of the WMAP data (Bennett & al 2003) has led us to a new era in observational cosmology. Yet, various claims for "unexpected" non-Gaussian signals cast a shadow on the purity of the data (Copi, Huterer & Starkman 2003; Eriksen & al 2004b; Coles & al 2003; Cruz & al 2003; Hansen & al 2004; Hansen, Banday & Górski 2004; Vielva & al 2003; Komatsu, Spergel & Wandelt 2003; Park 2003; Larson & Wandelt 2004; Mukherjee & Wang 2004; Eriksen & al 2004a; Schwarz & al 2004; Oliveira-Costa & al 2004). Given that there is no plausible theoretical explanation for these signals, a natural worry is that the maps are not, after all, fully free from systematics errors or galactic contamination.

Most notably several groups have reported evidence for a preferred axis being selected by large-angle fluctuations, either in the form of multipole planarity (Oliveira-Costa & al 2004; Copi, Huterer & Starkman 2003; Schwarz & al 2004; Coles & al 2003), North-South asymmetries in the power spectrum, three-point function, or bispectrum (Eriksen & al 2004a; Hansen, Banday & Górski 2004; Land & Magueijo 2004), as well as using other methods (Eriksen & al 2004b; Hansen & al 2004).

In assessing these asymmetries it is important to distinguish issues of non-Gaussianity (which should be rotationally invariant) from those of anisotropy (existence of a preferred axis). Apart from a subtlety in the definition of statistical ensemble (Ferreira & Magueijo 1997) these issues should be clearly separated. Unfortunately no *systematic* approach for extracting all the independent invariants under rotations from a given set of multipoles, $a_{\ell m}$, has been proposed. The formalism of Copi, Huterer & Starkman (2003) does not produce invariants. The invariant n-point correlation function (Magueijo 1995; Ferreira, Magueijo & Górski 1998; Magueijo 2000; Magueijo & Medeiros 2003), on the other hand, is awkward to apply and often contains redundant information.

In this letter we remedy this deficiency in the current formalism.

2 STATEMENT OF THE PROBLEM

Multipoles are irreducible representations of $SO(3)$, so their $2\ell+1$ degrees of freedom should split into $2\ell-2$ invariants and 3 rotational degrees of freedom. Ideally we would like to process the $\{a_{\ell m}\}$ for a given mutipole ℓ into the power spectrum C_ℓ (the Gaussian degree of freedom), $2\ell-3$ invariant measures of non-Gaussianity, and a system of axes (assessing isotropy). In addition one should build 3 invariants per multipole pair, for example the Euler angles relating the two sets of multipole axes. The latter encode inter-scale correlations.

No one has ever accomplished this project for general ℓ, but see Magueijo (1995) for a Quadrupole solution. Multipoles are equivalent to symmetric traceless tensors of rank ℓ. The problem is then to extract the independent invariant contractions of these tensors plus a system of axes – a generalisation of the concept of eigenvalues and eigen-

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vectors. Typically the invariants produced by this formalism are related to the $n-$point correlation function (bispectrum, trispectrum, etc.) The formalism becomes very complicated very quickly, and no systematic breakdown of independent $n-$point correlation function components has ever been achieved.

Alternatively one may extract from the $\{a_{\ell m}\}$ a length scale and ℓ independent unit vectors (the multipole vectors) as proposed by Schwarz & al (2004), and discussed further by Katz & Weeks (2004). This approach is considerably simpler and unsurprisingly it has been taken further. However we see a (correctable) problem with this approach. The multipole vectors are not rotationally invariant, and so they mix up the issues of isotropy and non-Gaussinity. We propose to correct this shortcoming by taking an appropriate number of independent inner products between these vectors. These are the sought-after invariants, and in some simple cases we relate them to the $n-$point correlation function. We will also extract from the multipole vectors a system of axes, encoding the multipole directional information. It is these axis that are to be used when testing isotropy.

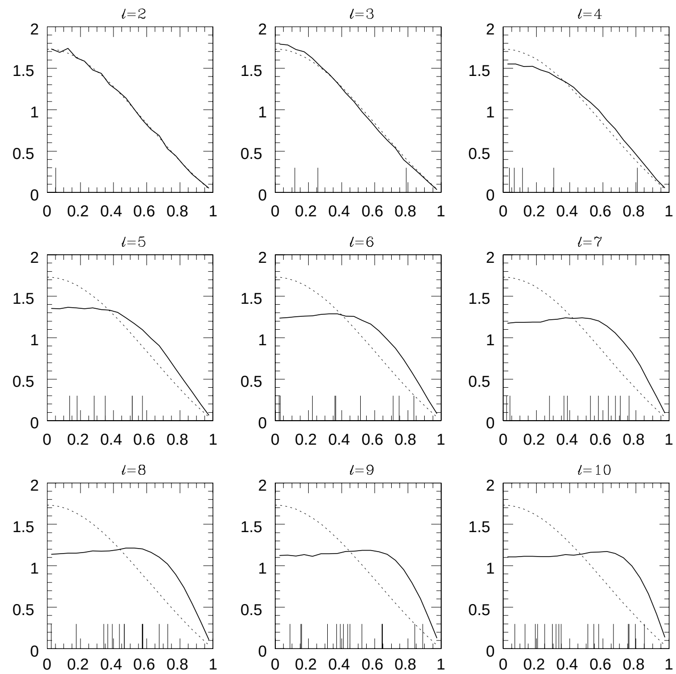

Figure 1. The average distribution of the modulus of the $2\ell-3$ dot products X_{ij} (with the two anchor vectors chosen at random) for multipoles $\ell=2-10$. The dotted line corresponds to the analytical expression found for the quadrupole. We have also plotted the invariant values for the WMAP first year data, with the anchor vectors chosen at random (short lines along the bottom of each panel).

3 THE EXAMPLE OF THE QUADRUPOLE

We start by considering the quadrupole, which as shown in Magueijo (1995) may be written as

$$\delta T_2 = Q_{ij}x^i x^j \tag{1}$$

where Q_{ij} is a symmetric traceless matrix and x^i are cartesian coordinates on the unit sphere. From this matrix one may extract three eigenvectors and two independent combinations of invariant eigenvalues λ_i. These are essentially the power spectrum C_2 (related to the sum of the squares of λ_i) and the bispectrum B_{222} (related to the determinant of the matrix, or the product of λ_i).

In contrast, following the formalism of Katz & Weeks (2004) one writes

$$\delta T_2 = A_2\Big(L^1_i L^2_j - \frac{1}{3}\delta_{ij}L^1\cdot L^2\Big)x^i x^j \tag{2}$$

where A is a scale and $\{L^1_i, L^2_i\}$ are two units vectors, encoding the information on non-Gaussianity and anisotropy.

The vectors $\{L^1_i, L^2_i\}$ are not invariant, but one can construct an invariant by taking $X = L^1\cdot L^2$. As pointed out to us by Starkman & Schwarz (2004), one may easily check that the eigenvectors of Q are

$$V^1 = L^1 + L^2 \tag{3}$$
$$V^2 = L^1 - L^2 \tag{4}$$
$$V^3 = L^1 \times L^2 \tag{5}$$

These have corresponding eigenvalues $A(X+3)/6$, $A(X-3)/6$, and $-AX/3$, respectively. Using results in Magueijo (1995) one may therefore prove that the power spectrum and bispectrum are:

$$C_2 = \frac{A_2^2}{6}(X^2+3) \tag{6}$$
$$B_{222} = \frac{A_2^3 X}{2^2 3^3}(9-X^2) \tag{7}$$

so that the normalised bispectrum is

$$I_2 = \frac{B_{222}}{C_2^{\frac{3}{2}}} = \frac{X(9-X^2)}{(X^2+3)^{3/2}} \tag{8}$$

These formulae bridge the two formalisms.

It was proved in Magueijo (1995) that for a Gaussian process C_2, I_2 and the eigenvectors are independent random variables, and that I_2 is uniformly distributed in the range $[-1,1]$ (and C_2 has a χ^2_5 distribution). We thus can prove that X is statistically independent from C_2 (but not from A) and from the eigenvectors. By directly evaluating the Jacobian of the transformation we find that its distribution is

$$P(X) = 27\frac{1-X^2}{(X^2+3)^{5/2}} \tag{9}$$

that is, it is not uniform. We have confirmed this result with Monte-Carlo simulations (see Fig. 1, top left panel).

This elucidates an interesting feature of multipole vectors. Even though they are algebraically independent (in the sense that they contain no redundant information given a concrete quadrupole realization) they are *not statistically independent*. If vectors L^1 and L^2 were statistically isotropic (which they are) and statistically independent, then X would be uniformly distributed. As (9) shows this is not the case; hence vectors L^1 and L^2 are statistically correlated. Specifically they prefer being orthogonal to being aligned.

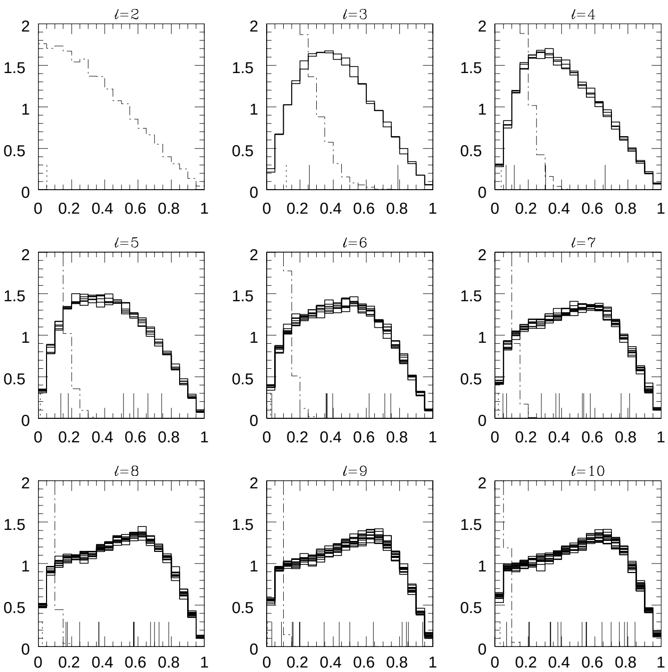

Figure 2. The distribution of the modulus of the dot products for multipoles $\ell = 2 - 10$, with anchor vectors L^1 and L^2 chosen as the most orthogonal pair of multipole vectors. The solid lines are for dot products X_{1i} and X_{2i}, and the dotted line shows the X_{12} distribution. The WMAP results are also plotted (short lines along the bottom, dashed line the X_{12} result).

4 THE GENERAL PROCEDURE

For a general multipole we have ℓ vectors and we could take as invariants all possible inner products between them. This would clearly lead to much redundant information, in contradiction with the requirements laid down in Section 2. A possible way out is to select two "anchor" vectors L^1 and L^2, consider their dot product $X_{12} = L^1 \cdot L^2$, and then for $3 \leqslant i \leqslant \ell$ the two products $X_{1i} = L^1 \cdot L^i$ and $X_{2i} = L^2 \cdot L^i$. We thus obtain $2\ell - 3$ algebraically independent invariants. In Fig. 1 we plot their distribution for $2 \leqslant \ell \leqslant 10$ after the order and direction ($\pm$) of the ℓ vectors has been randomised. The anchor vectors L^1 and L^2 are therefore selected at random, so that all X_{ij} invariants for a given multipole have the same distribution. We checked that the invariants X are uncorrelated. As we can see this distribution is ℓ dependent, and the tendency for multipole vectors to seek orthogonal directions is less pronounced for higher multipoles. We have also plotted these invariants as computed for the WMAP co-added masked first year map (details in Land & Magueijo (2004)). No evidence for non-Gaussianity is found, but we defer a closer scrutiny to a future publication.

The proposed procedure provides an interesting Gaussianity test. In particular it can be easily applied to extract all the relevant information for high ℓ; in contrast isolating such information from the n-point correlation function is most cumbersome. However the suggested algorithm suffers from the drawback that the anchor vectors L^1 and L^2 are selected at random, and so the procedure is not reproducible.

We may correct this by imposing a criterium for their selection, for example L^1 and L^2 may be taken as the most orthogonal among the given ℓ vectors. We may then form the "eigenvectors" (three orthogonal vectors):

$$V^1 = L^1 + L^2 \tag{10}$$
$$V^2 = L^1 - L^2 \tag{11}$$
$$V^3 = L^1 \times L^2 \tag{12}$$

as the natural variables for encoding the multipole directionality. For invariants we may take the power spectrum and the $2\ell - 3$ quantities:

$$X_{12} = L^1 \cdot L^2 \tag{13}$$
$$X_{1i} = L^1 \cdot L^i \tag{14}$$
$$X_{2i} = L^2 \cdot L^i \tag{15}$$

(for $3 \leqslant i \leqslant \ell$). These encode all the relevant non-Gaussian degrees of freedom.

The transformation from $a_{\ell m}$ into these variables is invertible (up to discrete uncertainties related to branch choice) and provides a solution to our problem, as phrased in Section 2. The proposed variables for a given multipole ℓ are the power spectrum C_ℓ (Gaussian degree of freedom), the $2\ell - 3$ inner products X (non-redundant non-Gaussian invariants), and the orthogonal vectors $\{V^1, V^2, V^3\}$ (measures of anisotropy). The procedure reduces to the one found for the quadrupole when $\ell = 2$. Within this framework the inter-ℓ correlations are measured by the Euler angles relating the systems of axes $\{V^1, V^2, V^3\}$ associated with each pair of multipoles. These should be uniformly distributed for a Gaussian distribution or indeed for any theory in which the various ℓ are uncorrelated.

In Fig. 2 we plot distributions for the X invariants with anchor vectors L^1 and L^2 defined as the most orthogonal. We have used 12,500 realizations to make these histograms. $P(X_{12})$ peaks around zero. The other X distributions are the same. We also plotted the invariants for the WMAP first year data, again finding no evidence for non-Gaussianity. We have checked that the eigenvectors, V^i, are uniformly distributed.

Notice that the invariants X cannot be independent variables, since their ranges of variation are interconnected. This is to be compared with the lack of independence among the various vectors L^i, as demonstrated in the previous Section.

Naturally we could have defined the two anchor vectors L^1 and L^2 in different ways, for example the two most aligned vectors. The invariant X_{12} would then be peaked around one. We plot the counterpart of Fig. 2 with this alternative definition in Fig. 3.

5 BRIDGING THE TWO FORMALISMS

For higher multipoles, relating the proposed formalism and the $n-$point correlation function, as done in Section 3 for the quadrupole, becomes very involved. For instance, for the octopole the counterpart of (1) and (2) is:

$$\delta T_3 = Q_{ijk} x^i x^j x^k = A_3 \Big(L^1_i L^2_j L^3_k - \frac{1}{3} \delta_{ij} R_k \Big) x^i x^j x^k \tag{16}$$

where one should note that terms in $\{i, j, k\}$ are symmetrized. The remainder R_k is:

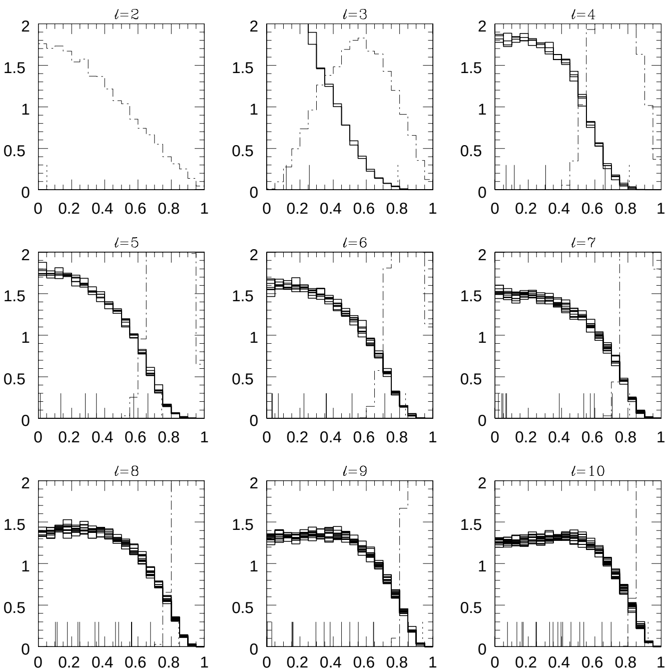

Figure 3. The distribution of the modulus of dot products for multipoles $\ell = 2 - 10$, with anchor vectors L^1 and L^2 chosen as the most aligned pair of multipole vectors. The solid lines are for dot products X_{1i} and X_{2i}, and the dotted line shows the X_{12} distribution. The WMAP results are also plotted (short lines along the bottom, dashed line the X_{12} result).

$$R_k = \frac{3}{5} L^{(1} \cdot L^2 L_k^{3)} \tag{17}$$

Again, $Tr(Q^2) = C_3$, and therefore one can show that:

$$C_3 = \frac{A_3^2}{6 \cdot 9}(9 + 7S_2 + 6S_2) \tag{18}$$

where

$$S_1 = (L^1 \cdot L^2)^2 + (L^2 \cdot L^3)^2 + (L^3 \cdot L^1)^2 \tag{19}$$
$$S_2 = (L^1 \cdot L^2) \times (L^2 \cdot L^3) \times (L^3 \cdot L^1) \tag{20}$$

For higher multipoles

$$Q_{i_1 \ldots i_\ell} = A_\ell \Big(L^{(1}_{i_1} \ldots L^{\ell)}_{i_\ell} - \frac{3}{5} \delta_{(i_1 i_2} (L^{(1} \cdot L^2 L^3_{i_3)} \ldots L^{\ell)}_{i_\ell} \Big) \tag{21}$$

The components of the n-point correlation function may be obtained from the various contractions of Q. They are clearly a function of the dot product of the various multipole vectors. However, the formulae become progressively more complex to derive. With the introduction of our alternative procedure we suggest that these complicated formulae are not necessary. The X variables provide a better basis for describing the data.

6 CONCLUSIONS

Harmonic (Fourier) space is the natural arena for comparing theory and observation for Gaussian theories. In several past studies it was also found to be a useful ground for testing the hypothesis of Gaussianity. In this paper we showed further that the degrees of freedom in the spherical harmonic components $\{a_{\ell m}\}$ can be simply separated into a set of algebraically independent invariants – the power spectrum and a set of $2\ell - 3$ non-Gaussian statistics – and a set of axes encoding the multipole directionality. The Euler angles relating sets of axes associated with pairs of multipoles measure inter-ℓ correlations. This provides an elegant answer to a long unsolved problem – how to process the information contained in a given map into its relevant non-redundant degrees of freedom.

Even though we have computed some of the proposed invariants for the WMAP first year map, we defer to a future publication a more systematic application of this framework to real data. We stress that the formalism is easily applicable for high ℓ multipoles. This is to be contrasted with the n-point correlation function. Even though it is trivial to evaluate the bispectrum at high ℓ (Land & Magueijo 2004) it becomes very difficult to distill all the non-redundant information contained in a given multipole in terms of sufficiently high order components of the correlation function. The proposed formalism is far more efficient.

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